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APPROXIMATE ANALYTIC SOLUTION OF A CYLINDRICAL IMPLODING SHOCK WAVE

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ABSTRACT

Due to increasing interest in the problem of implosion, this problem is viewed as the necessary step in realizing thermonuclear fusion. The imploding cylindrical shock wave is included by a cylindrical piston converging onto the central axis. The flow behind the imploding shock has been studied. Imploding shock propagating through homogeneous medium near the axis of implosion has been considered when the flow assumes self-similar character. An approximate analytic solution of the problem in a closed by employing the Chernyii's expansion technique has been obtained in which the flow variable is expanded in a series of powers of Ɛ the density ratio across the strong shock. An analytical expression for the similarity exponent λ from the singular point analysis of the single differential equation has also been obtained. The problem of imploding shock studied belong to a class of self – similar motion of the second kind. An approximate analytic solution of the flow behind the cylindrical imploding shock propagating in a homogeneous medium in a closed form has been obtained when the flow is adiabatic.

KEYWORDS: Implosion, Imploding Cylindrical Shock, Self-Similar, Homo-Thermal & Similarity Exponent

INTRODUCTION

There is an increased interest in the problem of implosion. Among other applications, it is viewed as a necessary step in realizing thermo-nuclear fusion, Mishkin, and Fujimoto[1]. Imploding shocks have been studied by Zel'dovich and Raizer[2], Mishkin and Fujimoto[1], Ashraf[3], Sachdev & Ashraf [4], Radha and Sharma[5] and others. The flow behind the imploding shock has been studied by several authors under both situations when the flow is adiabatic or homo-thermal ,i.e. When the temperature gradient behind the shock is zero. The imploding cylindrical shock wave can be thought of as induced by a cylindrical piston converging onto the central axis. The imploding shock or a sequence of such shocks, can compress substantially the matter around the axis of symmetry. Mishkin and Fujimoto[1] have studied the problem of a cylindrical shock by numerical methods and have obtained an analytic expression for the self – similarity exponent α from the consideration of single maximum pressure behind the shock.

In this paper, we consider imploding cylindrical shock propagating through a homogeneous medium near the axis of implosion when the flow assumes a self similar character. We assume the shock wave to move in a perfect gas with a constant specific heat ratio $\gamma = c_p/c_v$ to its axis at time t= 0. A strong imploding shock, or a sequence of such shocks, can compress substantially the matter around the axis of symmetry. We assume the gas to present a continuum and neglect its molecular structure and the dissipative processes associated with it. The problem is essentially the same as has been investigated by Mishkin and Fujimoto[1], who have obtained the solution numerically and have also obtained an analytic expression for the similarity exponent α from the considerations of single maximum pressure behind the shock. We have obtained an approximate analytic solution of the problem in a closed form by employing Chernyii's[6] technique in which the flow variables are expanded in a power series in term of ε, the density ratio across the strong shock. The imploding shock is an example of a problem belonging to the class of self – similar motion of the second kind in which the similarity exponent α, occurring in the law of shock propagation, cannot be determined in advance from the physical considerations ,but is determined by solving an Eigen- value problem for a single differential equation to which the similarity equations are reducible. We have found an analytic expression of the self –similarity co-efficient λ(γ), where $\alpha = \frac{1}{1-\lambda}$, from the considerations of singular points of the single differential equation.

BASIC EQUATIONS

The mass, momentum and energy conservation equations in cylindrical symmetry from following Landau and Lifshtiz[8] and Courant and Friedrichs[9] are

$$\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial r}+\rho\frac{u}{r}=0 \quad (2.1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\ \rho^{-1}\frac{\partial p}{\partial r}=0 \quad (2.2)$$

$$\frac{\partial(p\rho^{-\gamma})}{\partial t}=0 \quad (2.3)$$

here p, ρ and u denote the pressure, density of the gas assumed ideal with constant specific heat ratio $\gamma = c_p/c_v$ and r is the radial distance from the axis. We consider a shock discontinuity moving through the gas of density ρ_0 and denote it by R(t), the position of the shock front and by $\dot{R}(t)$, its velocity. In case of a very strong shock the pressure, the shock encounters can be neglected and thus the boundary conditions at the shock are

$$u_s=\frac{2}{\gamma+1}\dot{R},\ \rho_s=\frac{\gamma+1}{\gamma-1}\rho_0,\ p_s=\frac{2}{\gamma+1}\ \rho_0\dot{R}^2 \quad (2.4)$$

We introduce a similarity variable

$$\xi=\frac{r}{R(t)} \quad (2.5)$$

and seek the solution in the similarity form as

$$u=\frac{2}{\gamma+1}\dot{R}u_1(\xi) \quad (2.6)$$

$$\rho=\frac{\gamma+1}{\gamma-1}\ \rho_0 Q(\xi) \quad (2.7)$$

$$P=\frac{2}{\gamma+1}\ \rho_0\ \dot{R}^2\ P(\xi) \quad (2.8)$$

Where $u_1, P(\xi), Q(\xi)$ are the dimensionless reduced velocity, pressure and density respectively. At the shock front ξ = 1, and

$$P(1)=Q(1)=u_1(1)=1 \quad (2.9)$$

A transformation

$$u_1 \rightarrow U_1=\frac{\gamma+1}{2}\ (U+\xi) \quad (2.10)$$

Introduces the new non-dimensional velocity U,

$$u = \dot{R}(U+\xi) \tag{2.11}$$

and simplifies the hydrodynamic derivatives.

The self-similar solution (2.6) to (2.8) reduce the partial differential equations, equations (2.1) to (2.3), for conservation laws to a set of three ordinary differential equations, Mishkin and Fujimoto[1].

$$-\frac{1}{Q}\frac{dQ}{d\xi} = \frac{1}{U}\frac{dU}{d\xi} + \frac{1}{\xi} + \frac{2}{U} \tag{2.12}$$

$$-\frac{2(\gamma-1)}{(\gamma+1)^2}\frac{1}{Q}\frac{dP}{dt} = U\frac{dU}{d\xi} + \gamma\xi + (1+\lambda)U \tag{2.13}$$

$$-\frac{1}{P}\frac{dP}{d\xi} = \gamma\left[\frac{1}{U}\frac{dU}{d\xi} + \frac{1}{\xi} + \frac{2(\lambda+\gamma)}{U}\right] \tag{2.14}$$

where $\lambda = \frac{d\,ln\dot{R}}{d\,lnR}$ (2.15)

and ρ_0 is the assumed a constant.

Separation of variables t and ξ requires λ = constant and hence

$$R(t) = R_0\left(1-\frac{t}{t_c}\right)\alpha\,;\alpha = \frac{1}{1-\lambda} \tag{2.16}$$

where t_c is the time when taken the explosion occurred earlier and r = R_0 at time t = 0.

Following Chernyii's[5] technique, in which it is assumed that most of the mass behind the shock is concentrated near the shock, we expand the reduced velocity, pressure and density in a series in terms of ε, the density ratio across the strong shock.

$$U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} \tag{2.17}$$

$$P = P^{(0)} + \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} \tag{2.18}$$

$$Q = \frac{Q^{(0)}}{\varepsilon} + Q^{(1)} + \varepsilon Q^{(2)} \tag{2.19}$$

The boundary conditions at the shock are

$$U^{(0)} = -\varepsilon, U^{(1)} = U^{(2)} = 0 \tag{2.20}$$

$$P^{(0)} = 1, P^{(1)} = P^{(2)} = 0 \tag{2.21}$$

$$Q^{(0)} = \varepsilon, Q^{(1)} = Q^{(2)} = 0 \tag{2.22}$$

We substitute the expansion (2.17) to (2.19) into equations (2.12) to (2.14) and obtain the following equations for the zeroeth order term.

$$-U^{(0)}\frac{dQ^{(0)}}{d\varepsilon} = Q^{(0)}\frac{dU^{(0)}}{d\varepsilon} + \frac{Q^{(0)}U^{(0)}}{\varepsilon}\,2R^{(0)} \tag{2.23}$$

$$-2(\gamma-1)\frac{dP^{(0)}}{d\varepsilon} = (\gamma+1)^2\left[Q^{(0)}U^{(0)}\frac{dQ^{(0)}}{d\varepsilon} + Q^{(0)}\lambda\varepsilon + Q^{(0)}(1+\lambda)U^{(0)}\right] = 0 \tag{2.24}$$

$$-U^{(0)}\frac{dP^{(0)}}{d\varepsilon} = \gamma\left[P^{(0)}\frac{dQ^{(0)}}{d\varepsilon} + P^{(0)}\frac{U^{(0)}}{\varepsilon}\right] + (\lambda+\gamma)2P^{(0)} \tag{2.25}$$

Integrating equations (2.23) to (2.25) with boundary conditions (2.20) to (2.22), we obtain the solution as

$$U = U^{(0)} = L - M\sqrt{(\varepsilon + N)} \tag{2.26}$$

$$P = P^{(0)} = \frac{B(L - M\sqrt{(\varepsilon+N)})^{-s}}{\xi^{\gamma}} \cdot e^{\frac{4(\gamma+\lambda)\sqrt{\varepsilon+N}}{M}} \tag{2.27}$$

$$Q = Q^{(0)} = A(L - M\sqrt{\varepsilon+N})^{\frac{-(M^2-4L)}{M^2}} e^{\frac{4}{M}\sqrt{\varepsilon+M}} \tag{2.28}$$

where, L = $\frac{\varepsilon^2}{2(\varepsilon+\lambda+1)}, M = \sqrt{8L(1+\lambda)}, N = \frac{-L}{2(1+\lambda)}, A = \varepsilon(L - M\sqrt{\varepsilon+N})^{\frac{-(M^2-4L)}{M^2}} e^{-\frac{4}{M}\sqrt{\varepsilon+M}}$

S = $\gamma\frac{(M^2-4L)}{M^2} - \frac{4\lambda L}{M^2}$

B = $(L - M\sqrt{(1+N)})^S . e^{\frac{-4(\gamma+\lambda)\sqrt{1+N}}{M}}$

From equations (2.12) to (2.14) we obtain

$$\frac{dP}{d\xi} = (\gamma+1)^2U^2Q\frac{[-P\gamma\lambda\xi U^{-2} - (1+\lambda)P\gamma U^{-1} + \gamma P\xi^{-1}\ 2(\gamma+\lambda)PU^{-1}]}{2\gamma(\gamma-1)P - (\gamma+1)^2U^2Q} \tag{2.29}$$

The differential equation (2.29) possesses two singular points. In order the solution may be physically meaningful, the solution curve, besides satisfying the boundary conditions at the shock, passes on through the appropriate singular point. The solution is finite and single valued if both the numerator and the denominator of the differential equations (2.29) vanish simultaneously. This condition provides an analytic expression for λ as

$$\lambda = -\frac{\gamma(\gamma-1)}{(\gamma+1)(2\gamma-1)} \tag{2.30}$$

RESULTS AND DISCUSSIONS

We have obtained the appropriate analytic solution of the flow behind a cylindrical imploding shock, propagating in a homogeneous medium, in closed form when the flow is adiabatic to the zeroth order approximation. The differential equations of the first and second order approximations were too complicated to lend analytic solution. Equations (2.26) to (2.28) give the reduced particle velocity, pressure and density distributions behind the shock. The error in the solution is of 0(ε), which is small if γ is 0(1). The solution to the reduced flow variables have been depicted graphically in the figures 1, 2 and 3 for γ = $6/5, 7/5, and\ 5/3$. As pointed earlier in the self- similar motions of the second kind, to which this problem belongs, the similarity exponent α cannot be determined in advance from physical considerations, but is obtained by solving an Eigen-value problem for a single differential equation to which the similarity equations are reducible. Mishkin and Fujimoto[1] have determined an analytic expression for λ, the similarity co-efficient, from the consideration of single maximum pressure behind the shock. We have found an analytic expression for λ in equation (2.30) from the consideration of the singular points of differential equation (2.29).

Table 1 gives the values of similarity exponent α, obtained from the equation (2.30), Table 2 gives the values of α determined from the analytic expression for α, Mishkin and Fujimoto[1]. Table – 3 gives the values of α, Sachdev and Ashraf[4], when the flow behind the shock is homothermal flows, then for the adiabatic flows so that the shock velocity

$\dot{R}\alpha t^{\alpha-1}$ is larger in the former case that in the latter as the shock approaches the axis of implosion. This is due to the larger pressure gradient in the shock region behind the homothermal flow.

As the shock converges, the energy behind it becomes concentrated near the shock front as the temperature and pressure there tend to infinity. However, the dimensions of the self-similar region decease with time and the total energy concentrated in this region also decreases. Sachdev and Ashraf[4] have shown that in the homothermal flow the total energy in the region of self- similar flow decreases more rapidly with time in the adiabatic flow. The self-similar solution, however, is valid in a small region near the axis, to which all non-self similar solutions of this problem finally converge.

Value of α obtained from an analytic expression

Table 1

γ	α
1.1	0.958
11/9	0.922
9/7	0.907
7/5	0.885
5/3	0.848
3	0.769
$2 + 3^{\frac{1}{2}}$	0.750

Values of α obtained from the Analytic expression of Mishkin and Fujimoto[1]

Table 2

γ	α
1.1	0.847
11/9	0.838
9/7	0.835
7/5	0.828
5/3	0.814
3	0.767
$2 + 3^{\frac{1}{2}}$	0.750

Values of α for homothermal flow

Table 3

γ	α
6/5	0.65530
7/5	0.5965
5/3	0.5587

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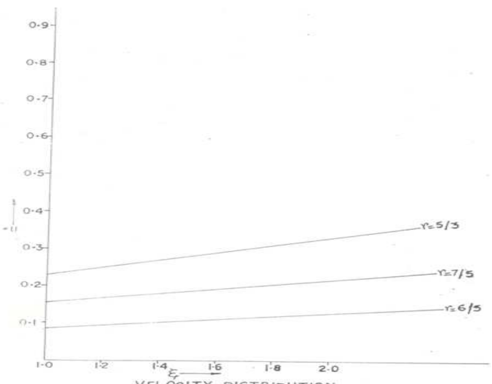
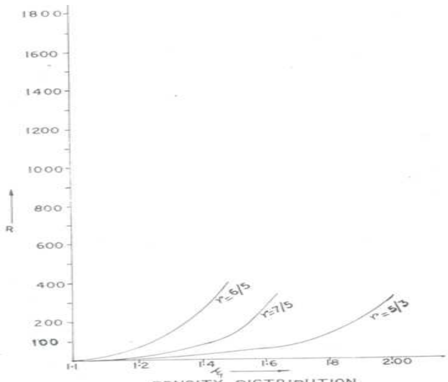
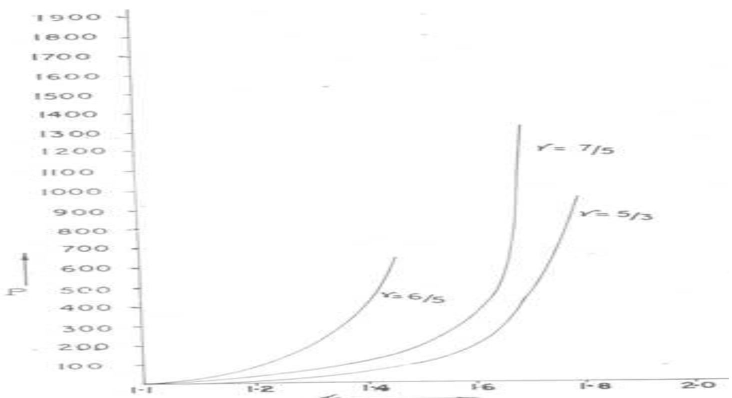


VELOCITY DISTRIBUTION
FIG. 1



DENSITY DISTRIBUTION
FIG. 2



PRESSURE DISTRIBUTION
FIG. 3