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ON THE APPLICATION OF ABOODH TRANSFORM TO SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this research, we have derived the general formulas for the solution of systems of partial differential equations with the help of the Aboodh integral transform. Various test examples have been deplored for the validation of the derived formulas, including homogeneous and nonhomogeneous systems with constant coefficients. Amazingly, the reported solutions are in perfect conformity with the literature.

KEYWORDS: Integral transforms, Aboodh transform, partial differential equations, system of differential equations.

1. INTRODUCTION

Differential equations and their applications continue to be enthusiastic in the fields of engineering and sciences owing to their immeasurable significance in understanding, as well as in modeling various phenomena. More so, the integral method of transformation from one domain to another is one of the oldest methods of the solution of differential equations, read the history and development of integral transforms like the Fourier and Laplace in [1]; of course, the Fourier and Laplace integral transforms are the oldest integral transforms ever devised. Additionally, researchers are still keen to the development of integral transforms and keep discovering dissimilar transforms with applications in all branches of knowledge. We mention the Sumudu integral transform [2,3] which happens to be among the most crucial integral transforms for tackling differential equations. Other integral transforms include Aboodh, Elzaki, Shehu, and Noval integral transforms [4,5,6,7], to state a few. In particular, Aboodh 2014 [8] proposed a new integral transform just like the Laplace to solve differential and integral equations via transformation. Besides, the application of this transform has been fruitfully extended to tackle various problems modeled as ordinary, partial, and fractional differential equations [10]. Thus, we give the recognized definition of the Aboodh integral transform of the function $U(t)$ as follows:

$$A[U(t),\eta] = K(\eta) = \frac{1}{\eta}\int_0^{\infty} U(t)\, e^{-t\eta}\, dt\,,\ t > 0, \tag{1.1}$$

such that

$$\eta \in (-\lambda_1, \lambda_2), \qquad \lambda_1\,, \lambda_2 > 0,$$

where $U(t)$ is a real function, $e^{-t\eta}$ is the kernel function, and A is the operator of the Aboodh transform; while η is the Aboodh transform parameter.

However, having stated a little about the development and application of the Aboodh transform, the present paper is aimed at extending the relevance of this integral transform to the complete treatment of the system of partial differential equations. Moreover, the fundamental aspects and assumptions of this change are discussed in the second Section. General formulas for the solution of systems of partial differential equations with constant coefficients are derived in the third Section. Finally, these formulas are used to solve several examples in the fourth Section; while the fifth Section is reserved for concluding remarks.

2. PRELIMINARIES

In this section, we give certain definitions, theorems and properties associated with the Aboodh integral transform. No doubt, this information would later be utilized in the course of establishing the aiming target of the present study.

2.1 Definition: [11]

The systems of first-order nonhomogeneous linear partial differential equations of interest in this study is of the form

$$\sum_{i=0}^{m}\sum_{j=0}^{n} U_{ij}\, k(x_1,\, x_2, \ldots, x_n)\, \frac{\partial U_i}{\partial x_j} = f(x) \quad k = 1,2,\ldots,r,$$

where $x_1, x_2, \ldots, x_n$ are independent variables (real or complex) and $U_1, U_2, \ldots, U_m$ are dependent variables and $U_{ij}k$ are given functions of class C^1.

2.2 Property: [12]

(i) If $U_1(x), U_2(x), \ldots, U_n(x)$ have Aboodh transform, then

$$A[\beta_1 U_1(x) + \beta_2 U_2(x) + \cdots + \beta_n U_n(x)] = \beta_1 A[U_1(x)] + \beta_2 A[U_2(x)] + \cdots + \beta_n A[U_n(x)],$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are constants $U_1(x), U_2(x), \ldots, U_n(x)$ are defined function.

(ii) If $A^{-1}[U_1(\eta)] = \gamma_1(t),\ A^{-1}[U_2(\eta)] = \gamma_2(t), \cdots, A^{-1}[U_n(\eta)] = \gamma_n(t)$, then

$$A^{-1}[\beta_1 U_1(\eta) + \beta_2 U_2(\eta) + \cdots + \beta_n U_n(\eta)] = \beta_1 \gamma_1(t) + \beta_2 \gamma_2(t) + \cdots + \beta_n \gamma_n(t),$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are constants, and A^{-1} is the Aboodh inverse transform operator.

2.3 Theorem: [9]

If the Aboodh transform of the function $U(t)$ is $K(x,\eta)$, then via integration by parts, we give the following results:

(i) $A\left[\frac{\partial U}{\partial t}(x,t)\right] = \eta K(x,\eta) - \frac{1}{\eta} U(x,0),$

(ii) $A\left[\frac{\partial^2 U}{\partial t^2}(x,t)\right] = \eta^2 K(x,\eta) - \frac{1}{\eta}\frac{\partial U(x,0)}{\partial t} - U(x,0),$

(iii) $A\left[\frac{\partial^n U}{\partial t^n}(x,t)\right] = \eta^n K(x,\eta) - \sum_{k=0}^{n-1} \frac{1}{\eta^{2-n+k}}\, U^{(k)}(x,0).$

Proof:

Proofs of (i) and (ii) follow effortlessly.

(iii) This result can easily be proven via mathematical induction. That is, for $n = 1$, (ii) gives

$A\left[\frac{\partial U}{\partial t}(x,t)\right] = \eta K(x,\eta) - \frac{1}{\eta}U(x,0),$

which is true. Next, it is supposed that the relation is true for $n = m$, that is

$A\left[\frac{\partial^m U}{\partial t^m}(x,t)\right] = \eta^m K(x,\eta) - \sum_{k=0}^{m-1} \frac{1}{\eta^{2-m+k}} U^{(k)}(x,0).$

Then, if $n = m + 1$, we can show that the relation

$A\left[U^{(m+1)}(x,t)\right] = \eta^{m+1} K(x,\eta) - \sum_{k=0}^{m} \frac{1}{\eta^{1-m+k}} U^{(k)}(x,0),$

is true.

Thus, it is enough put $U^{(m)}(x,t) = g(t)$, so we have

$A[U^{(m+1)}(x,t)] = A[g'(t)].$

and hence the result.

2.3 Remark: [13]

Using the Leibniz's rule to obtain the Aboodh transform for partial derivatives as follows:

(i) $A\left[\frac{\partial U(x,\eta)}{\partial x}\right] = \int_0^\infty e^{-t\eta} \frac{\partial U(x,\eta)}{\partial x} dt = \frac{\partial}{\partial x}\int_0^\infty e^{-t\eta} U(x,\eta) dt = \frac{\partial}{\partial x}\left[A[U(x,\eta)]\right] = \frac{d}{dx}[A[U(x,\eta)]],$

(ii) $A\left[\frac{\partial^2 U(x,\eta)}{\partial x^2}\right] = \int_0^\infty e^{-t\eta} \frac{\partial^2 U(x,\eta)}{\partial x^2} dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-t\eta} U(x,\eta) dt = \frac{\partial^2}{\partial x^2}\left[A[U(x,\eta)]\right] = \frac{d^2}{dx^2}[A[U(x,\eta)]],$

(iii) $A\left[\frac{\partial^n U(x,\eta)}{\partial x^n}\right] = \int_0^\infty e^{-t\eta} \frac{\partial^n U(x,\eta)}{\partial x^n} dt = \frac{\partial^n}{\partial x^n}\int_0^\infty e^{-t\eta} U(x,\eta) dt = \frac{\partial^n}{\partial x^n}\left[A[U(x,\eta)]\right] = \frac{d^n}{dx^n}[A[U(x,\eta)]].$

Additionally, Table 1 gives the Aboodh transform of certain prominent functions [14]. Moreover, there exist various studies with regard to the application of the Aboodh integral transform in diverse fields of concern, see [15-20] and the references therein.

Table 1: Aboodh Transform for Some Functions [14]

Function $U(t)$	$A[U(t)] = \frac{1}{\eta}\int_0^\infty e^{-\eta t} U(t) dt = K(\eta)$
1	$\frac{1}{\eta^2}$
$t^n,\ n > 0$	$\frac{n!}{\eta^{n+2}}$
e^{at}	$\frac{1}{\eta(\eta - a)}$
$\sinh(at)$	$\frac{a}{\eta(\eta^2 - a^2)}$
$\cosh(at)$	$\frac{1}{\eta^2 - a^2}$

3. MAIN RESULTS

This section presents the main results of the present study. That is, the general formulas of the solution for systems of partial differential equations would be derived by the application of the Aboodh integral transform.

3.1 Theorem 1

Let us consider the coupled system of first-order nonhomogeneous partial differential equation:

$$\left.\begin{array}{l} K_t(x,t)+U_x(x,t)=f_1(x,t) \\ U_t(x,t)+K_x(x,t)=f_2(x,t)\end{array}\right\}, \tag{3.1}$$

with the initial conditions $K(x,0)=\lambda_1(x)$, $U(x,0)=\lambda_2(x)$ and boundary conditions $K(0,t)=K(1,t)=0, U(0,t)=U(1,t)=0$. Then, by the application of Aboodh transform, the system admits the following solution closed-form solution:

$$U(x,t)=A^{-1}\left[\left[\left(\frac{1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\ \lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right]\right],$$

$$K(x,\eta)=\frac{1}{\eta}\ A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1(x)-\frac{1}{\eta}\frac{d}{dx}\left[\left(\frac{1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right]. \tag{3.2}$$

Proof: By applying the Aboodh transform to both sides of the coupled system given in equation (3.1), one gets:

$$\eta\, K(x,\eta)-\frac{1}{\eta}K(x,0)+\frac{d}{dx}\ U(x,\eta)=A\big(f_1(x,t)\big),$$

$$\eta\, U(x,\eta)-\frac{1}{\eta}U(x,0)+\frac{d}{dx}\ K(x,\eta)=A\big(f_2(x,t)\big).$$

Then, the above equations after substituting the prescribed initial conditions become:

$$\frac{d}{dx}\ U(x,\eta)+\eta K(x,\eta)=A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1(x), \tag{3.3}$$

$$\frac{d}{dx}\ K(x,\eta)+\eta\ U(x,\eta)=A\big(f_2(x,t)\big)+\frac{1}{\eta}\lambda_2(x). \tag{3.4}$$

Now, from equation (3.3), we get

$$\frac{d}{dx}K(x,\eta)=\frac{1}{\eta}\frac{d}{dx}\ A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1{}'(x)-\frac{1}{\eta}\frac{d^2}{dx^2}U(x,\eta). \tag{3.5}$$

Then, upon substituting equation (3.5) in equation (3.4), we get

$$\frac{d^2}{dx^2}U(x,\eta)-\eta^2U(x,\eta)=\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1{}'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x). \tag{3.6}$$

Therefore, equation (3.6) represents nonhomogeneous ordinary differential equation of order two that has the following solution form:

$$U(x,\eta)=U_c(x,\eta)+U_p(x,\eta),$$

where

$U_c(x,\eta) = B_1 e^{\eta x} + B_2 e^{-\eta x}$,

while on employing variation of parameters, other solution is obtained as follows:

$U_p(x,\eta) = B_1(x) e^{\eta x} + B_2(x) e^{-\eta x}$.

More so,

$B_1'(x) e^{\eta x} + B_2'(x) e^{-\eta x} = 0$,

$\eta B_1'(x) e^{\eta x} - \eta B_2'(x) e^{-\eta x} = \frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1{}' - \eta A\big(f_2(x,t)\big) - \lambda_2(x)$,

such that

$\Delta = \begin{vmatrix} e^{\eta x} & e^{-\eta x} \\ \eta e^{\eta x} & -\eta e^{-\eta x} \end{vmatrix} = -2\eta$,

$$B_1'(x) = \frac{1}{\Delta}\begin{vmatrix} 0 & e^{-\eta x} \\ \frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x) & -\eta e^{-\eta x} \end{vmatrix},$$

$$= \frac{\left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{-\eta x}}{2\eta},$$

or

$B_1(x) = \int \left[\frac{\left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{-\eta x}}{2\eta}\right] dx.$

In the same way, we obtain $B_2(x)$ as follows:

$B_2(x) = \int \left[\frac{\left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{\eta x}}{-2\eta}\right] dx.$

Consequently,

$$U(x,\eta) = B_1 e^{\eta x} + B_2 e^{-\eta x} + \left[\left(\frac{1}{2\eta}\int \left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{-\eta x}\, dx\right) e^{\eta x} + \left(\frac{-1}{2\eta}\int \left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{\eta x}\, dx\right) e^{-\eta x}\right],$$

after which upon deploying the prescribed boundary conditions reveals that $B_1 = B_2 = 0$, and subsequently gives the solution of $U(x,\eta)$ as follows:

$$U(x,\eta) = \left[\left(\frac{1}{2\eta}\int \left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{-\eta x}\, dx\right) e^{\eta x} + \left(\frac{-1}{2\eta}\int \left(\frac{d}{dx} A\big(f_1(x,t)\big) + \frac{1}{\eta}\lambda_1'(x) - \eta A\big(f_2(x,t)\big) - \lambda_2(x)\right) e^{\eta x}\, dx\right) e^{-\eta x}\right].$$

The solution of $K(x,\eta)$ is equally obtained in the same way as follows:

$$K(x,\eta)=\frac{1}{\eta}\ A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1(x)-\frac{1}{\eta}\frac{d}{dx}\Big[\Big(\frac{1}{2\eta}\int\Big(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\Big)\,e^{-\eta x}\,dx\Big)\,e^{\eta x}+\Big(\frac{-1}{2\eta}\int\Big(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\Big)\,e^{\eta x}\,dx\Big)\,e^{-\eta x}\Big].$$

Evidently, upon taking the inverse Aboodh transform of the above expressions for $U(x,\eta)$ and $K(x,\eta)$, one gets the results of Theorem 1.

3.2 Theorem 2

Let us consider the coupled system of second-order nonhomogeneous partial differential equation:

$$\left.\begin{aligned}K_{tt}(x,t)+U_x(x,t)=f_1(x,t)\\U_{tt}(x,t)+K_x(x,t)=f_2(x,t)\end{aligned}\right\},\tag{3.7}$$

with the initial conditions $K(x,0)=\lambda_1(x), U(x,0)=\lambda_2(x)$, and the boundary conditions $K_t(x,0)=\partial_1(x)$, $U_t(x,0)=\partial_2(x)$, $K(0,t)=K(1,t)=0, U(0,t)=U(1,t)=0$. Then, by the application of Aboodh transform, the system admits the following solution closed-form solution:

$$U(x,t)=A^{-1}\Bigg[\Bigg[\Bigg(\frac{1}{2\eta^2}\int\Bigg(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\Bigg)e^{-\eta^2x}\,dx\Bigg)e^{\eta^2x}+\Bigg(-\frac{1}{2\eta^2}\int\Bigg(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\Bigg)e^{\eta^2x}\,dx\Bigg)e^{-\eta^2x}\Bigg]\Bigg],$$

$$K(x,t)=A^{-1}\Bigg[\frac{1}{\eta^2}A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1\,(x)+\frac{1}{\eta^3}\partial_1(x)-\frac{1}{\eta^2}\frac{d}{dx}\Bigg[\Bigg(\frac{1}{2\eta^2}\int\Bigg(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\Bigg)e^{-\eta^2x}\,dx\Bigg)e^{\eta^2x}+\Bigg(-\frac{1}{2\eta^2}\int\Bigg(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\Bigg)e^{\eta^2x}\,dx\Bigg)e^{-\eta^2x}\Bigg]\Bigg].\tag{3.8}$$

Proof: applying the Aboodh transform to the governing model in equation (3.7), one gets:

$$\eta^2\,K(x,\eta)-K(x,0)-\frac{1}{\eta}\,K_t(x,0)+\frac{d}{dx}\,U(x,\eta)=A\big(f_1(x,t)\big),$$

$$\eta^2\,U(x,\eta)-U(x,0)-\frac{1}{\eta}U_t(x,0)+\frac{d}{dx}\,K(x,\eta)=A\big(f_2(x,t)\big),$$

of which after substituting the initial conditions becomes:

$$\frac{d}{dx}U(x,\eta)+\eta^2K(x,\eta)=A\big(f_1(x,t)\big)+\lambda_1(x)+\frac{1}{\eta}\partial_1(x),\tag{3.9}$$

$$\frac{d}{dx}K(x,\eta)+\eta^2U(x,\eta)=A\big(f_2(x,t)\big)+\lambda_2(x)+\frac{1}{\eta}\partial_2(x).\tag{3.10}$$

Next, simple calculation from equation (3.9) gives:

$$\frac{d}{dx}K(x,\eta) = \frac{d}{dx}\frac{1}{\eta^2}A\big(f_1(x,t)\big) + \frac{1}{\eta^2}\lambda_1{}'(x) + \frac{1}{\eta^3}\partial'_1(x) - \frac{1}{\eta^2}\frac{d^2}{dx^2}U(x,\eta). \quad (3.11)$$

Then, substituting equation (3.11) in equation (3.10) reveals:

$$\frac{d^2}{dx^2}U(x,\eta) - \eta^4 U(x,\eta) = \frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1{}'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\partial_2(x). \quad (3.12)$$

The equation (3.12) represents a nonhomogeneous ordinary differential equation of order two that admits the following solution form:

$$U(x,\eta) = U_c(x,\eta) + U_p(x,\eta),$$

where

$$U_c(x,\eta) = B_1 e^{\eta^2 x} + B_2 e^{-\eta^2 x},$$

while U_p can be obtained through:

$$U_p(x,\eta) = B_1(x)e^{\eta^2 x} + B_2(x)e^{-\eta^2 x}.$$

Now, since

$$B_1'(x)e^{\eta^2 x} + B_2'(x)e^{-\eta^2 x}(x) = 0,$$

$$\eta^2 B_1'(x)e^{\eta^2 x} - \eta^2 B_2'(x)e^{-\eta^2 x} = \frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1{}'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x),$$

where

$$\Delta = \begin{vmatrix} e^{\eta^2 x} & e^{-\eta^2 x} \\ \eta^2 e^{\eta^2 x} & -\eta^2 e^{\eta^2 x} \end{vmatrix} = -2\eta^2.$$

Thus, as in the preceding case, we get $B_1(x)$ and $B_2(x)$ as follows

$$B_1(x) = \frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1{}'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx,$$

$$B_2(x) = -\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1{}'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx.$$

Moreover, the first solution pair $U(x,\eta)$ takes the following form

$$U(x,\eta) = B_1 e^{\eta^2 x} + B_2 e^{-\eta^2 x} + \left[\left(\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} + \left(-\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

where the application of the boundary data gives $B_1 = B_2 = 0$, such that the pair $U(x,\eta)$ becomes:

$$U(x,\eta)=\left[\left(\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\right)e^{-\eta^2x}\,dx\right)e^{\eta^2x}+\left(-\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\right)e^{\eta^2x}\,dx\right)e^{-\eta^2x}\right].$$

Moreover, in similar way, other solution pair $K(x,\eta)$ could be obtained as follows:

$$K(x,\eta)=\frac{1}{\eta^2}A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1(x)+\frac{1}{\eta^3}\partial_1(x)-\frac{1}{\eta^2}\frac{d}{dx}\left[\left(\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\right)e^{-\eta^2x}\,dx\right)e^{\eta^2x}+\left(-\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\lambda_1'(x)+\frac{1}{\eta}\partial'_1(x)-\eta^2A\big(f_2(x,t)\big)-\eta^2\lambda_2(x)-\eta\,\partial_2(x)\right)e^{\eta^2x}\,dx\right)e^{-\eta^2x}\right].$$

Finally, upon taking the inverse Aboodh transform of the above expression, the theorem becomes evident.

4. APPLICATIONS

The utility and effectiveness of the Aboodh transform method on the coupled systems of partial differential equations are demonstrated in the present section. Moreover, the derived solution forms in Theorems 1 and 2 in the above section will be utilized to obtain exact solutions of certain test problems.

Example 1

Consider the homogeneous system of partial differential equations as follows

$$\left.\begin{aligned}K_t(x,t)+U_x(x,t)=0\\U_t(x,t)+K_x(x,t)=0\end{aligned}\right\}, \tag{4.1}$$

with the initial and boundary conditions $K(x,0)=e^{5x}$, $U(x,0)=e^{-5x}$, $K(0,t)=K(1,t)=0, U(0,t)=U(1,t)=0$.

Without further delay, the first solution pair $U(x,t)$ takes the following form

$$U(x,t)=A^{-1}\left[\left(\frac{1}{2\eta}\int\left(\frac{1}{\eta}\lambda_1'(x)-\lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{1}{\eta}\lambda_1'(x)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right],$$

$$=A^{-1}\left[\left(\frac{1}{2\eta}\int\left(\frac{5}{\eta}e^{5x}-e^{-5x}\right)e^{-\eta x}\,dx\right)e^{\eta x}-\left(\frac{1}{2\eta}\int\left(\frac{5}{\eta}e^{5x}-e^{-5x}\right)e^{\eta x}\,dx\right)e^{-\eta x}\right],$$

$$=A^{-1}\left[\left(\frac{1}{2\eta}\int\frac{5}{\eta}e^{(5-\eta)x}\,dx\right)e^{\eta x}-\left(\frac{1}{2\eta}\int e^{(-5-\eta)x}\,dx\right)e^{\eta x}-\left(\frac{1}{2\eta}\int\frac{5}{\eta}e^{(5+\eta)x}\,dx\right)e^{-\eta x}-\left(\frac{1}{2\eta}\int e^{(-5+\eta)x}\,dx\right)e^{-\eta x}\right],$$

$$=A^{-1}\left[\frac{5}{2\eta^2}\cdot\frac{1}{5-\eta}e^{5x}-\frac{1}{2\eta}\cdot\frac{1}{-5-\eta}e^{-5x}-\frac{5}{2\eta^2}\cdot\frac{1}{5+\eta}e^{5x}-\frac{1}{2\eta}\cdot\frac{1}{-5+\eta}e^{-5x}\right],$$

$$=A^{-1}\left[\left(\frac{-5}{\eta(\eta^2-25)}e^{5x}+\frac{1}{\eta^2-25}e^{-5x}\right)\right].$$

$$\therefore U(x,t)=-e^{5x}\sinh(5t)+e^{-5x}\cosh(5t).$$

Moreover, for the second solution pair $K(x,t)$, one obtains

$$K(x,t)=A^{-1}\left[\frac{1}{\eta^2}\lambda_1(x)-\frac{1}{\eta}\frac{d}{dx}\left[\left(\frac{1}{2\eta}\int\left(\frac{1}{\eta}\lambda_1'(x)-\lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{1}{\eta}\lambda_1'(x)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right]\right],$$

$$=A^{-1}\left[\frac{1}{\eta^2}e^{5x}-\frac{1}{\eta}\frac{d}{dx}\left[\frac{-5}{\eta(\eta^2-25)}e^{5x}+\frac{1}{\eta^2-25}e^{-5x}\right]\right],$$

$$=A^{-1}\left[\frac{1}{\eta^2}e^{5x}+\frac{25}{\eta^2(\eta^2-25)}e^{5x}+\frac{5}{\eta(\eta^2-25)}e^{-5x}\right],$$

$$=A^{-1}\left[\frac{1}{\eta^2-25}e^{5x}+\frac{5}{\eta(\eta^2-25)}e^{-5x}\right].$$

$$\therefore K(x,t)=e^{5x}\cosh(5t)+e^{-5x}\sinh(5t)\,.$$

Example 2

Consider the nonhomogeneous system of partial differential equations as follows

$$\left.\begin{aligned}U_x(x,t)+K_t(x,t)&=3x^2\\K_x(x,t)+U_t(x,t)&=-12t\end{aligned}\right\},\qquad (4.2)$$

with the initial and boundary conditions $K(x,0)=0, U(x,0)=-12$ and $K(0,t)=K(1,t)=0, U(0,t)=U(1,t)=0$.

As proceeded, the governing model admits the following solution via Theorem 1. Thus, the solution pair $U(x,t)$ takes the form:

$$U(x,t)=A^{-1}\left[\left(\frac{1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right],$$

$$=A^{-1}\left[\left(\frac{1}{2\eta}\int\left(\frac{6x}{\eta^2}+\frac{12\eta}{\eta^3}+12\right)e^{-\eta x}\,dx\right)e^{\eta x}-\left(\frac{1}{2\eta}\int\left(\frac{6x}{\eta^2}+\frac{12\eta}{\eta^3}+12\right)e^{\eta x}\,dx\right)e^{-\eta x}\right],$$

$$=A^{-1}\left[\left(\frac{3}{\eta^3}\int xe^{-\eta x}\,dx+\frac{6}{\eta^3}\int e^{-\eta x}\,dx+\frac{6}{\eta}\int e^{-\eta x}\,dx\right)e^{\eta x}-\left(\frac{3}{\eta^3}\int xe^{\eta x}\,dx+\frac{6}{\eta^3}\int e^{\eta x}\,dx+\frac{6}{\eta}\int e^{\eta x}\,dx\right)e^{-\eta x}\right],$$

$$=\mathrm{A}^{-1}\left[\frac{-6\mathrm{x}}{\eta^4}-\frac{12}{\eta^4}-\frac{12}{\eta^2}\right],$$

$$\therefore U(x,t)=-3xt^2-6t^2-12\,.$$

Moreover, for the second solution pair $K(x,t)$ takes the following form:

$$K(x,t)=A^{-1}\left[\frac{1}{\eta}A\big(f_1(x,t)\big)+\frac{1}{\eta^2}\lambda_1(x)-\frac{1}{\eta}\frac{d}{dx}\left[\left(\frac{1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{-\eta x}\,dx\right)e^{\eta x}+\left(\frac{-1}{2\eta}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big)+\frac{1}{\eta}\lambda_1'(x)-\eta A\big(f_2(x,t)\big)-\lambda_2(x)\right)e^{\eta x}\,dx\right)e^{-\eta x}\right]\right]$$

$$=A^{-1}\left[\frac{3x^2}{\eta^3}-\frac{1}{\eta}\frac{d}{dx}\left[\frac{-6x}{\eta^4}-\frac{12}{\eta^4}-\frac{12}{\eta^2}\right]\right]=A^{-1}\left[\frac{3x^2}{\eta^3}-\frac{1}{\eta}\left[\frac{-6}{\eta^4}\right]\right]=A^{-1}\left[\frac{3x^2}{\eta^3}+\frac{6}{\eta^5}\right].$$

$\therefore K(x,t) = 3x^2t + t^3$.

Example 3

Consider the homogeneous system of partial differential equations as follows

$$\left.\begin{matrix} U_{tt}(x,t) + K_x(x,t) = 0 \\ K_{tt}(x,t) + U_x(x,t) = 0 \end{matrix}\right\}, \qquad (4.3)$$

with the initial and boundary conditions $U(x,0) = 4x, K(x,0) = -4x,\ U_t(x,0) = 4, K_t(x,0) = -4,\ U(0,t) = U(1,t) = 0, K(0,t) = K(1,t) = 0$.

As proceeded, the governing model admits the following solution via Theorem 2 with $U(x,t)$ admitting the following expression

$$U(x,t) = A^{-1}\left[\left(\frac{1}{2\eta^2}\int\left(\lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} + \left(-\frac{1}{2\eta^2}\int\left(\lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

$$= A^{-1}\left[\left(\frac{1}{2\eta^2}\int(4 + 4x\eta^2 + 4\eta)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} - \left(\frac{1}{2\eta^2}\int(4 + 4x\eta^2 + 4\eta)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

$$= A^{-1}\left[\frac{-4}{\eta^4} - \frac{4x}{\eta^2} - \frac{4}{\eta^3}\right],$$

$\therefore U(x,t) = -2t^2 - 4x - 4t.$

Also, for the second pair $K(x,t)$, we get:

$$K(x,t) = A^{-1}\left[\frac{1}{\eta^2}\lambda_1(x) + \frac{1}{\eta^3}\partial_1(x) - \frac{1}{\eta^2}\frac{d}{dx}\left[\left(\frac{1}{2\eta^2}\int\left(\lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} + \left(-\frac{1}{2\eta^2}\int\left(\lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right]\right],$$

$$= A^{-1}\left[\frac{4x}{\eta^2} + \frac{4}{\eta^3} - \frac{1}{\eta^2}\frac{d}{dx}\left[\frac{-4}{\eta^4} - \frac{4x}{\eta^2} - \frac{4}{\eta^3}\right]\right] = A^{-1}\left[\frac{4x}{\eta^2} + \frac{4}{\eta^3} - \frac{1}{\eta^2}\left[\frac{-4}{\eta^2}\right]\right],$$

$$= A^{-1}\left[\frac{4x}{\eta^2} + \frac{4}{\eta^3} + \frac{4}{\eta^4}\right].$$

$\therefore K(x,t) = 4x + 4t + 2t^2$.

Example 4

Consider the nonhomogeneous system of partial differential equations as follows

$$\left.\begin{matrix} K_{tt}(x,t) + U_x(x,t) = 0 \\ U_{tt}(x,t) + K_x(x,t) = 2t - 2 \end{matrix}\right\}, \qquad (4.4)$$

with the following initial and boundary conditions $U(x,0) = 2x, K(x,0) = -2x,\ U_t(x,0) = 2, K_t(x,0) = -2$, $U(0,t) = U(1,t) = 0, K(0,t) = K(1,t) = 0$.

Therefore, upon using Theorem 2, the solution follows:

$$U(x,t) = A^{-1}\left[\left(\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} + \left(-\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

$$= A^{-1}\left[\left(\frac{1}{2\eta^2}\int\left(2 - \eta^2\left(\frac{2}{\eta^3} - \frac{2}{\eta^2}\right) + 2x\eta^2 + 2\eta\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} - \left(\frac{1}{2\eta^2}\int\left(2 - \eta^2\left(\frac{2}{\eta^3} - \frac{2}{\eta^2}\right) + 2x\eta^2 + 2\eta\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

$$= A^{-1}\left[\left(\frac{1}{\eta^2}\int e^{-\eta^2 x}\,dx - \frac{1}{\eta^3}\int e^{-\eta^2 x}\,dx + \frac{1}{\eta^2}\int e^{-\eta^2 x}\,dx + \int xe^{-\eta^2 x}\,dx + \frac{1}{\eta}\int e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} - \left(\frac{1}{\eta^2}\int e^{\eta^2 x}\,dx - \frac{1}{\eta^3}\int e^{\eta^2 x}\,dx + \frac{1}{\eta^2}\int e^{\eta^2 x}\,dx + \int xe^{\eta^2 x}\,dx + \frac{1}{\eta}\int e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right],$$

$$= A^{-1}\left[-\frac{4}{\eta^4} + \frac{2}{\eta^5} - \frac{2x}{\eta^2} - \frac{2}{\eta^3}\right],$$

$$\therefore U(x,t) = -2t^2 + \frac{1}{3}t^3 - 2x - 2t.$$

More so, we get $K(x,t)$ as follows:

$$K(x,t) = A^{-1}\left[\frac{1}{\eta^2}A\big(f_1(x,t)\big) + \frac{1}{\eta^2}\lambda_1(x) + \frac{1}{\eta^3}\partial_1(x) - \frac{1}{\eta^2}\frac{d}{dx}\left[\left(\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{-\eta^2 x}\,dx\right)e^{\eta^2 x} + \left(-\frac{1}{2\eta^2}\int\left(\frac{d}{dx}A\big(f_1(x,t)\big) + \lambda_1'(x) + \frac{1}{\eta}\partial'_1(x) - \eta^2 A\big(f_2(x,t)\big) - \eta^2\lambda_2(x) - \eta\,\partial_2(x)\right)e^{\eta^2 x}\,dx\right)e^{-\eta^2 x}\right]\right],$$

$$= A^{-1}\left[\frac{2x}{\eta^2} + \frac{2}{\eta^3} - \frac{1}{\eta^2}\frac{d}{dx}\left[-\frac{4}{\eta^4} + \frac{2}{\eta^5} - \frac{2x}{\eta^2} - \frac{2}{\eta^3}\right]\right],$$

$$= A^{-1}\left[\frac{2x}{\eta^2} + \frac{2}{\eta^3} - \frac{1}{\eta^2}\left[-\frac{2}{\eta^2}\right]\right],$$

$$= A^{-1}\left[\frac{2x}{\eta^2} + \frac{2}{\eta^3} + \frac{2}{\eta^4}\right].$$

$$\therefore K(x,t) = 2x + 2t + t^2.$$

CONCLUSIONS

In this paper, the present study makes use of the new integral transform called the Aboodh integral transform to derive solution formulas for a coupled system of partial differential equations. Various test examples have been deplored for the validation of the derived formulas, including homogeneous and nonhomogeneous systems with constant coefficients. More so, the reported solutions are in perfect conformity with the literature. Conclusively, the proposed method is very practical, straightforward, and can be applied to different systems of differential equations like higher-order and nonlinear models (via the decomposition approach).

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